An Analytical Study on Prediction of Effective Elastic Constants of Perforated Plate

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In this study, the validity of the Eshelby-type model for predicting the effective Young's modulus and in-plane Poisson's ratio of the 2-dimensional perforated plate has been investigated in terms of the porosity size and its arrangement. The predicted results by the Eshelby-type model are compared with those by finite element analysis. Whenever the ratio of the porosity size to the specimen size becomes smaller than 0.07, the effective elastic constants predicted by finite element analysis are convergent regardless of the arrangement of the porosities. Under these conditions, the effective Young's moduli of the perforated plate can be predicted within the accuracy of 5% by the Eshelby-type model, which overestimates and underestimates the effective Poisson's ratios by 10% and 6% for the plates with periodically and non-periodically arranged porosities, respectively.

Key Words : Perforated Plate, Effective Elastic Constants, Eshelby Method, Finite Element Analysis, Porosity Size, Porosity Arrangement

Nomenclature

λ : Lame's constant
μ : Lame's constant
ν : Poisson's ratio
σ^o : Applied stress
Ω : Porosities domain
$D-\Omega$: Solid domain
C : Stiffness
d : Diameter of the porosity
E : Young's modulus of the solid
e^o : Strain field induced in the solid without porosities by σ^o
$\bar{e}$: Average disturbance of the strain field in the solid
e : Disturbed strain field in the porosities
e^* : Eigenstrain field of Eshelby's equivalent inclusion method
e^c : Total strain field of the perforated plate
e^m : Total strain field in the solid
e^p : Total strain field in the porosities
f : Volume fraction of porosities
L : Length of the perforated plate
N : Number of the porosities
S : Eshelby tensor
W : Width of the perforated plate

Superscripts

c : The perforated plate
m : Solid
p : Porosities

1. Introduction

Porous solids have been widely used and investigated for biomedical applications, energy absorption materials and piezoelectric actuators. The porous NiTi alloys have the inherent characteristics of biocompatibility, reduced weight, and superelasticity. (Entchev and Lagoudas, 2004) The deformation mechanism of the porous aluminum alloy has been changed from brittle fracture

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to ductile deformation, resulting in the increase of energy absorption capacity. (Ryu et al., 2003) Porous piezoelectric ceramics such as lead zirconate titanate offer significant improvements over solid piezoelectric ceramics in many piezoelectric transducer design figures of merit. (Ting, 1985)

Many researches have been conducted for predicting mechanical and electromechanical properties of the porous materials. The material constants of perforated plate have been computed by applying standard finite element method to an unit cell, which is the periodically repeated minimum structure of the plate. (Lee, 1995 ; Chung, 2004) The thermo-mechanical behavior of the porous shape memory alloy has been estimated with the unit cell finite element method and an averaging micromechanics method for periodic distribution and random distribution of pores in shape memory alloy, respectively. (Qidwai et al., 2001) Eshelby-type models have been widely used for predicting the effective material properties of elastic and piezoelectric porous materials. (Tandon and Weng, 1986 ; Zhao et al., 1989 ; Dunn and Taya, 1993 ; Lee and Kim, 2005) These two kinds of method, based on the finite element method and the Eshelby-type model, have been extensively used to determine the effective material properties of the porous materials. The applicability of the former is restricted to the porous materials containing periodically distributed pores. In contrast, it is known that the latter is a more versatile method than the former, because it can be applied to the porous materials containing both periodically and randomly distributed pores and requires less complicated computational works.

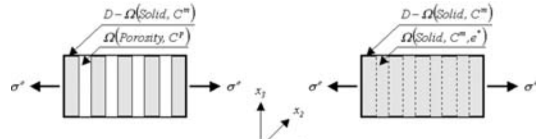
Eshelby-type models have been applied to predict the effective material properties of the composite due to their relative simplicity. As porous materials have been highlighted, many researchers have been greatly interested in Eshelby-type models to estimate the effective thermo-mechanical and electro-mechanical behaviors of porous materials. (Zhao et al., 1989 ; Dunn and Taya, 1993 ; Wu, 2000 ; Qidwai et al., 2001) However, any research on limitation of this type model has not been made in terms of pore size and its arrangement.

In the present study, the effective elastic constants of the perforated plate with the constant volume fraction of the porosities are derived explicitly by Eshelby's equivalent inclusion method (Eshelby, 1957) combined with Mori-Tanaka's mean field theory. (Mori and Tanaka, 1973) These analytical results are compared with the numerical results by finite element analysis. The model plate for the analysis is made of aluminum with the porosities of cylindrical shape, where the porosities of different sizes are distributed periodically and non-periodically. The applicability of the Eshelby-type model will be in detail discussed throughout this research in terms of the pore size and its arrangement.

2. Formulation

The 2-dimensional perforated plate is simulated for the analytical study as shown in Fig. 1, where the plate is uniformly loaded in x_1 direction. The original problem is schematically shown in Fig. 1(a), which is converted into the Eshelby's equivalent inclusion problem as shown in Fig. 1(b). $D-\Omega$ and Ω represent the isotropic solid and porosity domains of the perforated plate, respectively, which hereafter are denoted as superscripts m and p, respectively. All subscripts take on the values 1, 2, 3, and repeated indices are summed over the same values unless stated otherwise.

In the absence of any porosity, the corresponding strain field e^o due to the applied stress σ^o would be given by $\sigma^o_{ij}=C^m_{ijkl}e^o_{kl}$, where



(a) (b)

Fig. 1 An analytical model for computing the effective material properties of the perforated plate, (a) original problem, which is converted to (b) Eshelby's equivalent inclusion problem

$$C_{ijkl}^{m}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{kj}) \tag{1}$$

and λ, μ, C_{ijkl}^{m}, and δ_{ij} are Lame's constants and the stiffness tensor of the solid, and the Kronecker delta, respectively. When the plate is subjected to the uniform stress, the average stress in the solid and porosity can be determined with the help of Eshelby's equivalent inclusion method (Eshelby, 1957) and the Mori-Tanaka mean field theory. (Mori and Tanaka, 1973) Since the porosity cannot carry any load, the average stress inside the porosities is zero and can be expressed as

$$\begin{aligned}\sigma_{ij}^{p}&=C_{ijkl}^{p}(e_{kl}^{o}+\bar{e}_{kl}+e_{kl})\\&=C_{ijkl}^{m}(e_{kl}^{o}+\bar{e}_{kl}+e_{kl}-e_{kl}^{*})=0\end{aligned} \tag{2}$$

where C, $\bar{e}$, e, e^{*} represent the stiffness, the average elastic strain in the solid domain, the strain disturbed by the existence of the inhomogeneity, and the equivalent eigenstrain of the equivalent inclusion, respectively. The average stress in the solid is expressed as

$$\sigma_{ij}^{m}=C_{ijkl}^{m}(e_{kl}^{o}+\bar{e}_{kl}) \tag{3}$$

From Eqs. (2)-(3) and the requirement that the integration of disturbed stress over the entire domain must vanish, $\bar{e}_{ij}$ is given as

$$\bar{e}_{ij}+f(e_{ij}-e_{ij}^{*})=0 \tag{4}$$

where f is the volume fraction of the porosities. The disturbed strain e in the porosity is related through Eshelby's tensor S as follows:

$$e_{ij}=S_{ijkl}e_{kl}^{*} \tag{5}$$

where S is functions of the Poisson's ratio of the solid and the shape of the porosity.

From Eqs. (1), (2), (4), and (5), the eigenstrain e^{*} is explicitly represented as

$$e_{11}^{*}=\frac{3-2\nu^{2}}{1-f}e_{11}^{o} \tag{6a}$$

$$e_{22}^{*}=-\frac{1-2\nu^{2}}{1-f}e_{11}^{o} \tag{6b}$$

The total strains in the solid and porosity, e^{m} and e^{p}, are given by

$$e_{ij}^{m}=e_{ij}^{o}+\bar{e}_{ij} \tag{7a}$$

$$e_{ij}^{p}=e_{ij}^{o}+\bar{e}_{ij}+e_{ij} \tag{7b}$$

The volume average of the strain induced in the perforated plate is computed by using Eqs. (4) and (7), and is expressed as

$$e_{ij}^{c}=e_{kl}^{o}+fe_{ki}^{*} \tag{8}$$

By inserting Eq. (6) into Eq. (8), the strains in x_1 and x_2 directions due to the uniform applied stress are explicitly derived as

$$e_{11}^{c}=\left[1+\frac{f}{1-f}(3-2\nu^{2})\right]\frac{\sigma^{o}}{E} \tag{9a}$$

$$e_{22}^{c}=-\left[\nu+\frac{f}{1-f}(1-2\nu^{2})\right]\frac{\sigma^{o}}{E} \tag{9b}$$

From Eq. (9), the effective Young's modulus and in-plane Poisson's ratio of the perforated plate are expressed in explicit form as

$$E_{11}^{c}=\frac{E}{1+\frac{f}{1-f}(3-2\nu^{2})} \tag{10}$$

$$\nu_{12}^{c}=\frac{\nu+\frac{f}{1+f}(1-2\nu^{2})}{1+\frac{f}{1-f}(3-2\nu^{2})} \tag{11}$$

3. Numerical Experiments

The model shown in Fig. 2 is considered to simulate numerically the effective elastic constants of the 2-dimensional perforated plate using the commercial finite element software, ANSYS, where the porosities of different sizes are distributed periodically and non-periodically. The model is selected to be bigger than the perforated plate itself to minimize the end effect known by Saint-Venant's principle. The porosities included in the plate are penny-shaped cylinder and material properties are summarized in Table 1. The volume fraction of the porosities is defined as the ratio of the porosity volume to the specimen volume and is represented as

Table 1 Material properties of the solid and porosity for analytical studies

	Aluminum	Porosity
Young's modulus	70 GPa	0
Poisson's ratio	0.33	0
Aspect ratio	–	∞
Volume fraction of porosities	–	0.2

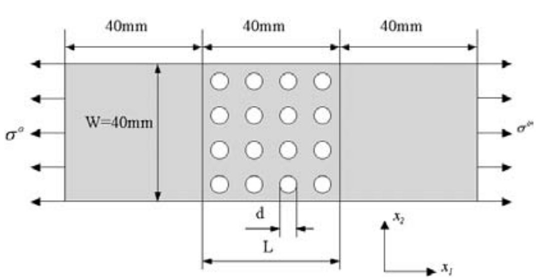
$$f=\frac{V_{porosity}}{V_{specimen}}=\frac{(\pi d^2/4)\,tN}{LWt} \tag{12}$$

where N, d, L, W, and t represent the number, the diameter of the porosities, the length, width, and thickness of the perforated plate, respectively. The diameter of the porosity is varied under its constant volume fraction of 0.2 to investigate the effect of the porosity size on the effective elastic constants of the plate. Since the diameter of the porosity is determined by the number of the porosity, Eq. (12) is reduced to

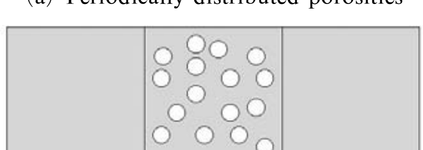
$$d=\sqrt{\frac{4fLW}{\pi N}} \tag{13}$$

The porosity is periodically distributed as shown in Fig. 2(a), where the porosity is located at the center of the unit rectangle. For the non-periodic distribution of the porosities, random numbers for a given diameter of the porosity are generated at least three times by Matlab program, which are used for the coordinates of centers of the porosities. Once the porosities are overlapped a little bit, the last one is excluded.

The displacement of the plate is computed by ANSYS software. The maximum, minimum, and average displacements of the plate with periodical distribution of the porosities are computed, while only the average displacement of the plate with non-periodic distribution of the porosities is computed. The effective elastic constants of the perforated plate under the condition shown in Fig. 2 are obtained are computed from the computed displacement by ANSYS, which are compared and discussed with those by Eshelby-type model depicted in chapter 2.



(a) Periodically distributed porosities

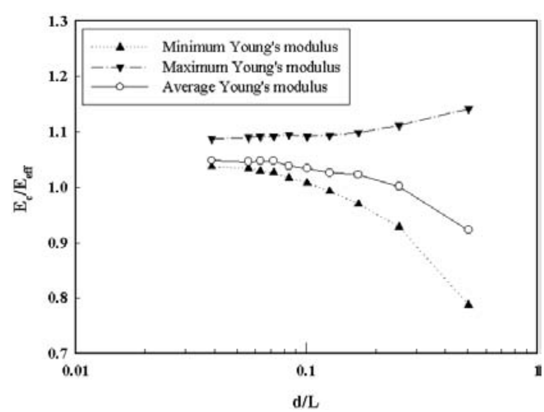


(b) Non-periodically distributed porosities

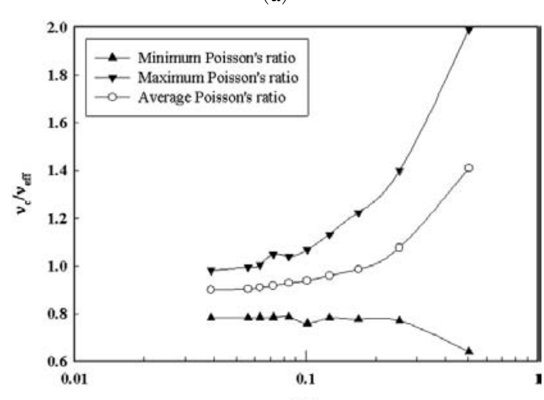
Fig. 2 Models of the 2-dimensional perforated plate with porosities of different numbers and distributions for a finite element analysis

4. Results and Discussions

Figure 3 shows the non-dimensional effective Young's modulus (E_c/E_{eff}) and in-plane Poisson's ratio (ν_c/ν_{eff}) of the perforated plate with



(a)



(b)

Fig. 3 Non-dimensionalized numerical results for effective elastic constants of 2-dimensional perforated plate with periodically distributed porosities as a function of porosity size: (a) effective Young's modulus, (b) effective Poisson's ratio the constant volume fraction of the periodically distributed porosities as a function of non-dimensional porosity size (d/L). The effective properties computed from the finite element analysis are non-dimensionalized by those derived from the Eshelby-type model (ETM) depicted in Chapter 2. The effective Young's modulus in direction of the applied stress and Poisson's ratio in the 1-2 plane are computed and their minimum, maximum, and average are done, too.

When the porosity is big, the minimum, maximum, and average values of the effective Young's modulus of the plate show considerable differences between them as shown in Fig. 3(a). At the largest d/L of 0.5 under the investigation, they are scattered from 0.8 to 1.15. However, they converge rapidly each other with decreasing the porosity size. As the porosity size becomes smaller than d/L of 0.07, they show the nearly constant discrepancies within about 8%. As far as the average Young's modulus is concerned, the ETM overestimates and underestimates the effective Young's moduli of the plates with larger and smaller sizes of porosities, respectively. For smaller size of porosities less than d/L of 0.07, the ETM predicts the average effective Young's modulus within the accuracy of about 5%. It can be concluded that the ETM predicts reasonably well the average effective Young's modulus of the plate with smaller porosities than d/L of 0.07.

As shown in Fig. 3(b), the effective Poisson's ratio of the perforated plate shows the similar trend to its effective Young's modulus in Fig. 3 (a). For the plate with smaller porosities than d/L of 0.07, the effective Poisson's ratio predicted by the ETM shows the maximum difference of 22% and differs from the average effective Poisson's ratio by about 10%, compared with the numerical results by FEM.

The average effective Young's modulus and Poisson's ratio of the perforated plate with the non-periodically distributed porosities are non-dimensionalized and plotted in Fig. 4. They are widely scattered at larger d/L, but their dispersion has diminished with decreasing the porosity size. For smaller porosities than d/L of 0.07, the average Young's modulus and Poisson's ratio takes almost constant values of 0.96 and 1.06, respectively. These numerical experiments confirm that the ETM can predict reasonably well the average elastic constants.

The validity of the ETM with respect to the distribution of the porosities is investigated and the results are plotted in Fig. 5, where the average effective elastic constants of the plates with periodically and non-periodically distributed porosities are compared. The effective Young's modulus of the plate with any distribution of the porosities by FEM analysis increases with decreasing the

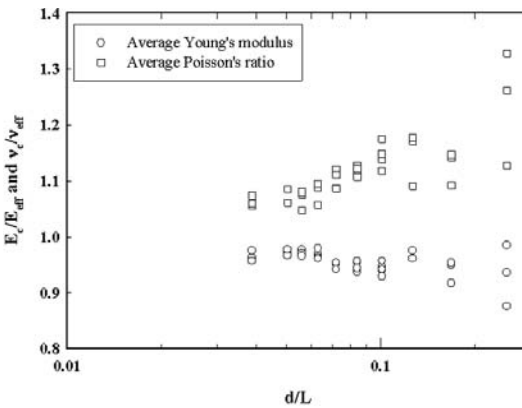


Fig. 4 Non-dimensionalized numerical results for effective elastic constants of 2-dimensional perforated plate with non-periodically distributed porosities as a function of porosity size

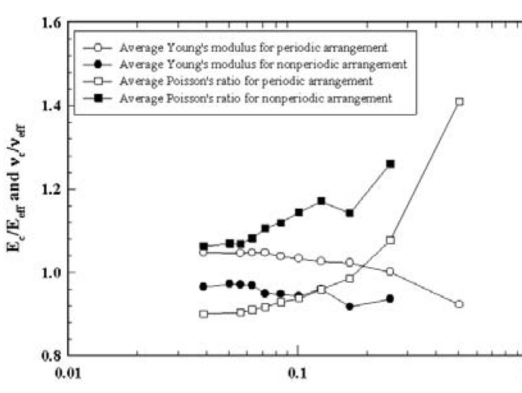


Fig. 5 Non-dimensionalized numerical results for average elastic constants of 2-dimensional perforated plate as functions of porosity size and its distribution

porosity size. The effective Young's modulus of the plate with the periodic porosities is larger by about 10% over the whole range of d/L than that with the non-periodic porosities. The ETM underestimates and overestimates by the same extent of at most 5% the effective Young's moduli of the plates with periodic and non-periodic porosities of the smallest size investigated, respectively. For non-periodically distributed porosities, the porosities are subject to be randomly distributed, which may induce high stress concentration. The higher stress concentration increases the area of large deformation, resulting in smaller Young's modulus of the plate. For smaller porosities than d/L of 0.07, the effective Young's moduli for both arrangements converges within about 8% difference. This indicates that the ETM can predict the effective Young's modulus quite accurately regardless of the distribution of the porosity.

As the porosity size decreases, the average effective Poisson's ratios of the plates with both distributions decrease rapidly. For non-periodic distribution of the porosities being less than d/L of 0.07, the effective Poisson's ratio predicted by ETM shows good agreement with the results by FEM analysis. Under the same conditions and non-periodic distribution of the porosities, the ETM overestimates the effective Poisson's ratio by about 10%. It can be concluded that the ETM can be applied to any distribution of the porosities and is well suited to the non-periodic distribution of the porosities for predicting the effective Poission's ratio.

5. Conclusions

The validity of the Eshelby-type model for predicting the effective elastic moduli of the 2-dimensional plate with the constant volume fraction of the porosities has been investigated in terms of the porosity size and its distribution. For this purpose, the predicted results by the Eshelby-type model are compared with those by finite element analysis. For the porosity size being smaller than d/L of 0.07 and any distribution of the porosities, it has been observed that the effective elastic constants predicted by finite element analysis have a rapid convergent trend. Under these conditions, the average effective Young's moduli of the plate can be accurately predicted with the accuracy of 5% by the Eshelby-type model. However, it overestimates and underestimates the average effective Poisson's ratios by 10% and 6% for the plates with periodically and non-periodically distributed porosities, respectively. As well noted, the Eshelby-type model has been proved to be more suitable for the plate with the non-periodically distributed porosities. It can be concluded through this study that the Eshelby-type model can predict relatively well the average effective elastic constants of the plate with both periodic and non-periodic distributions of the porosities and the smaller porosity size than d/L of 0.07.

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